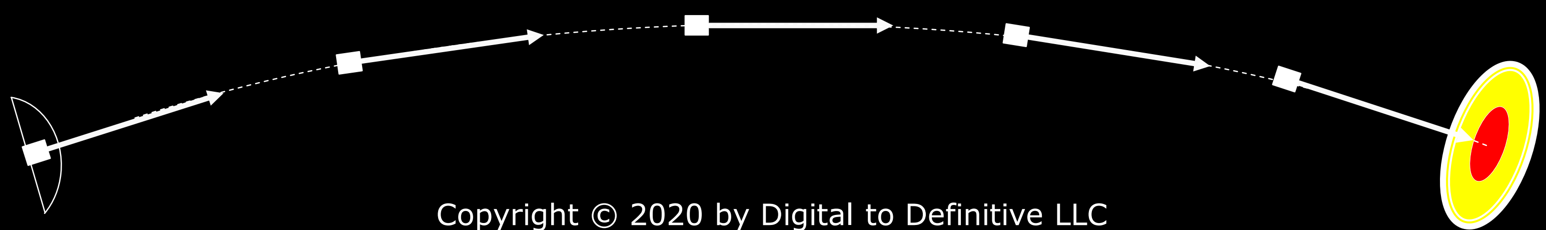
Engineering Insights on Archery

Arrow Launch and Flight



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As Presented to the Ashby Bowhunting Foundation
August 22, 2020

Darrel Barnette
Digital to Definitive (DtD)

Mandate

"You should spare neither effort nor expense in achieving perfect arrow flight. Even with every other factor in place, without good arrow flight you will have poor arrow performance"

-Dr. Ed Ashby

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Background of Presenter

Like many, I like to shoot, hunt, and enjoy the outdoors,

4-C

TEXARKANA GAZETTE

Local team wins state gun crown

The Texarkana Jaycee Air Gun team won the Arkansas State Championship June 7, at West Memphis, Ark., by scoring 1,864 points out of a possible 2,000.

They beat the second place team, Rogers, Ark., by a comfortable margin of 78 points. It was the third consecutive year for the local Texarkana team to win the Arkansas title. The team will now compete in the 10th annual Daisy-U.S. Jaycees International B.B. Gun Championships to be held in Clarksville, Tennessee in July.

Members of the winning team are twins, Kevin and Lowell McBryde, their sister, Sonya McBryde, Scott Powell, Darrel Barnette and alternate Coy Murray. Murray was also the high alternate at the state match.

The local championship team will now advance to the national finals to be held in July, which will feature 48 other state champions, including teams from Canada and Mexico.

Besides the test and shooting, other educational activities at the International includes a barter bar trade, a trip to Fort Campbell, Kentucky, a country cookout, country and western show featuring Tommy Cash, a trip to Opryland at Nashville.

Staff photo

STATE CHAMPIONS—These six members of the Texarkana Jaycee Air Gun team recently captured the Arkansas State championship in the state shoot in West Memphis as they pose with their trophies. Members of the team are kneeling: Kevin and Lowell McBryde; back row, l-r: Scott Powell, Darrel Barnette, Sonya McBryde and Coy Murray.

but, I am neither an archer nor a bow hunter (yet). However…

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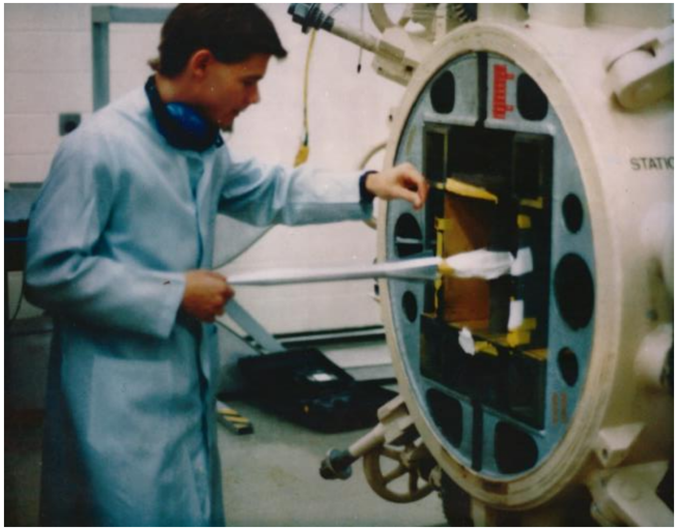
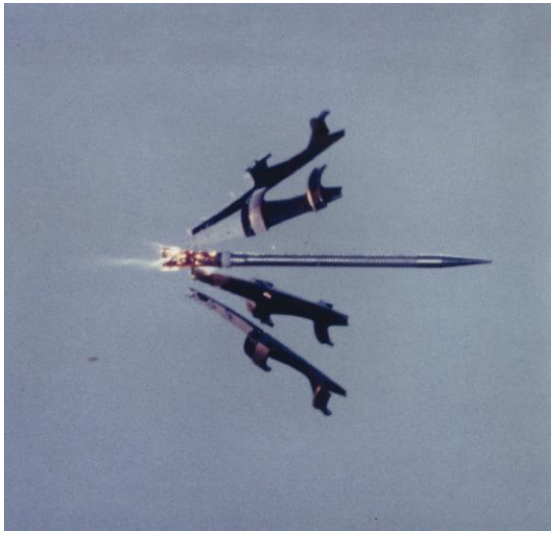
Background of Presenter

…I've spent 32 years in the DoD scientific community, working in applied aerodynamics and ballistics.





While I cannot speak with authority about specific equipment, in this presentation I hope to provide underlying principles and insights of arrow flight that you can use to make smart decisions about your bow hunting equipment.

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Presentation Topics

- **Aerodynamic Characteristics of Idealized vs Actual Arrow Flight**
- **The Relationship between Kinetic Energy, Momentum, and Retardation**
- **Questions and Discussion**

Presentation Topics

Part 1: The Kinetics of Arrow Flight

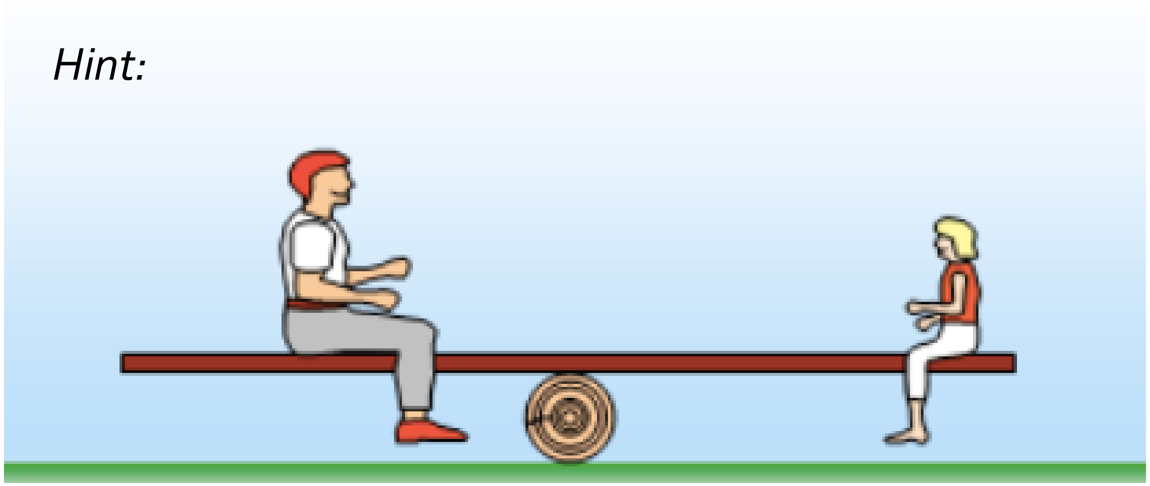
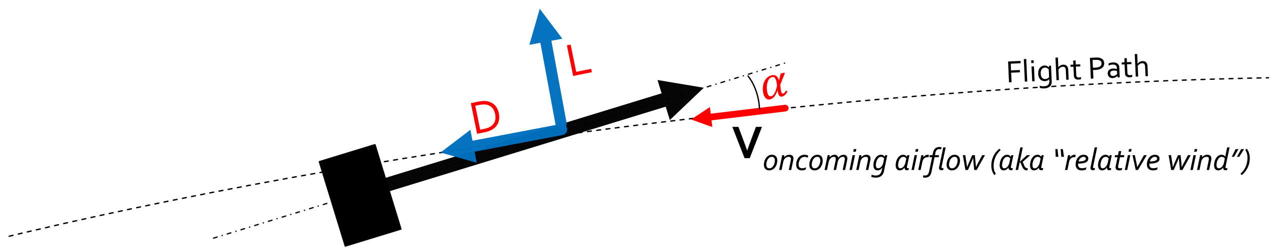


Image from: https://siobhannixon.wordpress.com/2016/12/08/the-physics-behind-a-seesaw/

Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Without loss of fidelity, we can pick a reference frame attached to an arrow, such that at any instant, the arrow is stationary and the air is flowing around it.



Wind Axis System

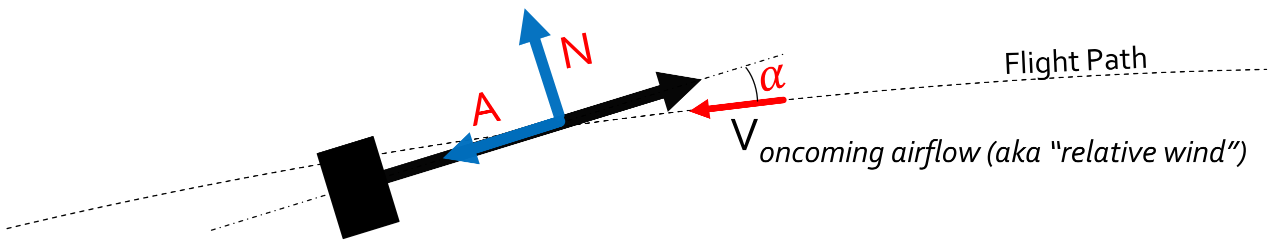
The basic aerodynamic forces acting on the arrow are the Lift Force (L), and the Drag Force (D).

- The angle α is called the angle of attack. It is defined as the angle between the arrow shaft axis and the local flight path.
- The Drag Force (D) acts parallel with the local flight path.
- The Lift Force (L) acts perpendicular to the local flight path and to the direction of (D).
- This coordinate system is referred to as the Wind Axis system as the Drag Force (D) acts in the direction of the oncoming airflow.

* Airplane Aerodynamicists typically use the Wind Axis system as it is intuitive that Drag opposes engine Thrust, and Lift generally opposes Weight.

Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Body Axis System:
An alternative to the Wind Axis Coordinate System can be described as follows:



- As before, the angle of attack, α is the angle between the arrow shaft axis and the local flight path.
- The Axial Force (A) acts along the longitudinal axis of the flight body.
- The Normal Force (N) acts perpendicular to the longitudinal axis of the flight body.

* Aeroballisticians typically prefer the Body Axis system as it is intuitive that the Normal Force (N) is a transverse force that can act in any radial direction.

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Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

The Wind Axis system (Lift Force, Drag Force) is commonly used to analyze the flight of Aircraft.

The Body Axis system (Normal Force, Axial Force) is commonly used to analyze the flight of Rotationally Symmetric Bodies such as Bombs, Missiles, Rockets, (and Arrows).

Even so, it is rather easy to transfer from one system to the other. For instance:

Given L, D:

$$\boldsymbol{N} = \boldsymbol{L}\cos(\boldsymbol{\alpha}) + \boldsymbol{D}\sin(\boldsymbol{\alpha})$$

$$\boldsymbol{A} = -\boldsymbol{L}\sin(\boldsymbol{\alpha}) + \boldsymbol{D}\cos(\boldsymbol{\alpha})$$

Given N, A:

$$\boldsymbol{L} = \boldsymbol{N}\cos(\boldsymbol{\alpha}) - \boldsymbol{A}\sin(\boldsymbol{\alpha})$$

$$\mathbf{D} = \boldsymbol{N}\sin(\boldsymbol{\alpha}) + \boldsymbol{A}\cos(\boldsymbol{\alpha})$$

Note that when $\boldsymbol{\alpha} = \mathbf{0}$: $\boldsymbol{L} = \boldsymbol{N},\ \boldsymbol{D} = \boldsymbol{A}$

http://www.aerospaceweb.org/question/aerodynamics/q0194.shtml

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Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Contextualization of the two systems:

Wind Axis:

"I hope the wings on this airplane can generate enough <u>lift</u> to keep us from falling out of the sky if the engine quits."

Body Axis:

"The initial tail-left yaw of the arrow generated enough <u>normal force</u> to push the impact point to the right of the bullseye."

We will use the Body Axis system for our Discussion Today

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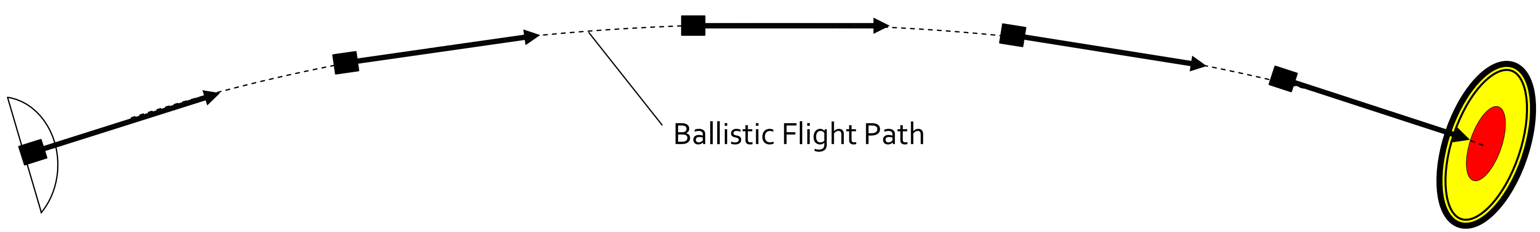
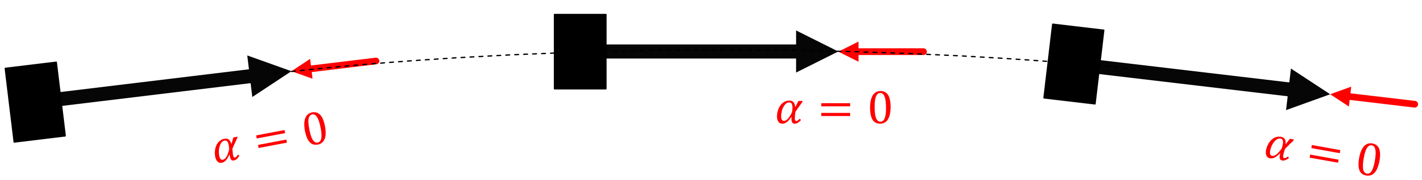
Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Perfect Arrow Flight:
In a perfect world, the arrow travels a ballistic trajectory where the only forces acting on the arrow are Weight and the opposing Axial Force.

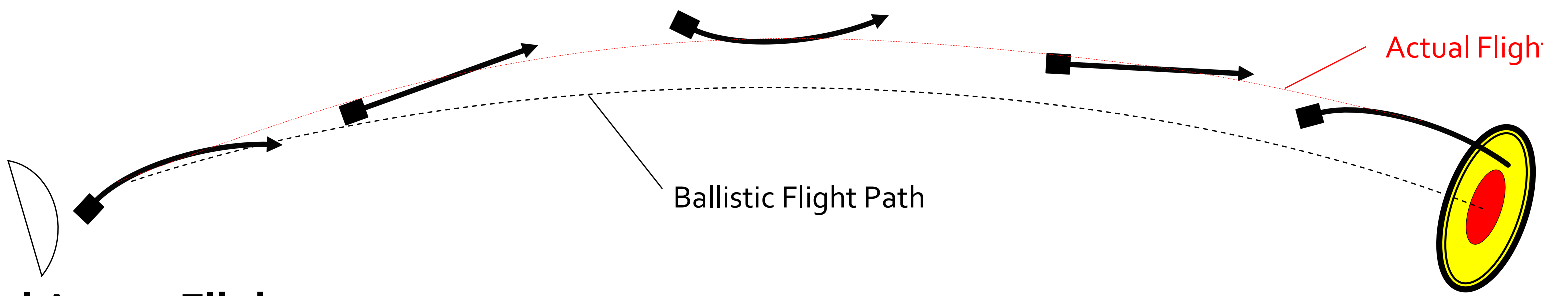


In a Ballistic Trajectory, the Normal Force doesn't occur because the arrow shaft remains ever-aligned with the local flight path, and therefore angle of attack, α is identically zero throughout the flight:



Olympic archers come very close to achieving ballistic flight with specialized equipment. See for example: T.Miyazaki, et al., *Aerodynamic properties of an archery arrow*, Sports Engineering, **16**, 43 – 45 (2013), available online.

Aerodynamic Characteristics of Perfect vs Actual Arrow Flight



Actual Arrow Flight:

- In reality, at a minimum an arrow bends, yaws, rolls, swerves, and precesses in flight.
- <u>Mechanical Asymmetries</u> and/or <u>User-Induced Errors</u> cause the arrow to fly with a constantly-changing, non-zero angle of attack with respect to the oncoming airflow.
- Any time a non-zero angle of attack is produced, a Normal Force is also produced.
- The presence of a Normal Force always affects the flight path of the arrow.

Recall Dr. Ashby's Mandate. To answer the mandate, for best arrow flight we must minimize the effect of disturbances and <u>mimic the ballistic flight trajectory of the arrow</u> as closely as possible.

Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Question: So is what you are saying that the presence of a Normal Force is the enemy of good arrow flight?

Answer: Emphatically, ***NO.*** The Normal Force is THE MUST-HAVE Aerodynamic Force Component Required for Arrow Stability.

Comment: There seems to be no standard reference for the aerodynamics associated with arrow flight. We instead must turn to Model Rocketry (especially as developed in the 1960's prior to the desktop computer age) for insight. For more information, refer to the listed references that follow.

Here's How the Normal Force controls arrow stability…

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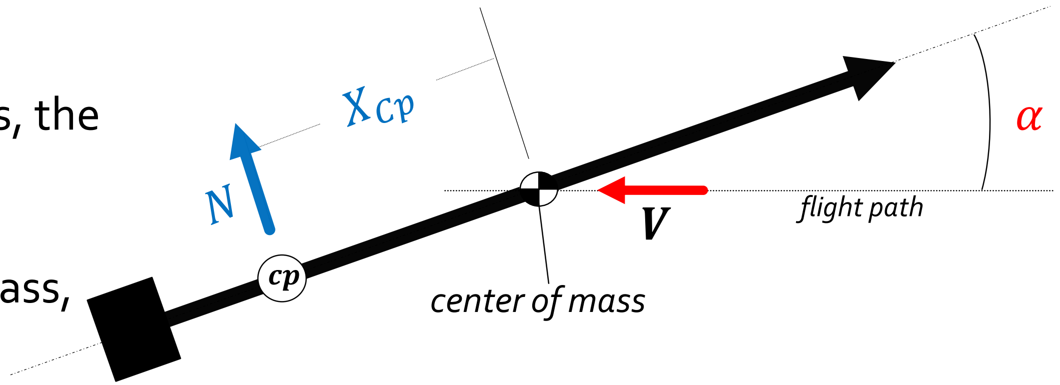
Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

- An Arrow in flight yaws about its center of mass.
- Whenever the angle of attack $\alpha > 0$, the pressure distribution around the arrow becomes asymmetric. The normal force generated by this pressure distribution can be represented by one force component **N**, acting at some distance **Xcp** away from the center of mass.

- If **Xcp** is behind the center of mass, the arrow is stable.

- If **Xcp** is in front of the center of mass, the arrow is unstable.



The use of a single summed Normal Force N acting at Xcp, is the classic text book depiction of stability for fin-stabilized projectiles, including arrows.

<u>We can cast this more intuitively for archers as follows…</u>

Chang H. S., *Rocket Stability and the Barrowman Equations, Midwest Rocketry Club, Jan. 2020*
https://westrocketry.com/www-test/index.php/articles/rocketry-stability-and-the-barrowman-equations/
Von Braun, W. *Why Rockets Have Fins*, Popular Science, Sept. 1964
https://books.google.com/books?id=MiYDAAAAMBAJ&lpg=PP1&pg=PA68&hl=en#v=onepage&q&f=false

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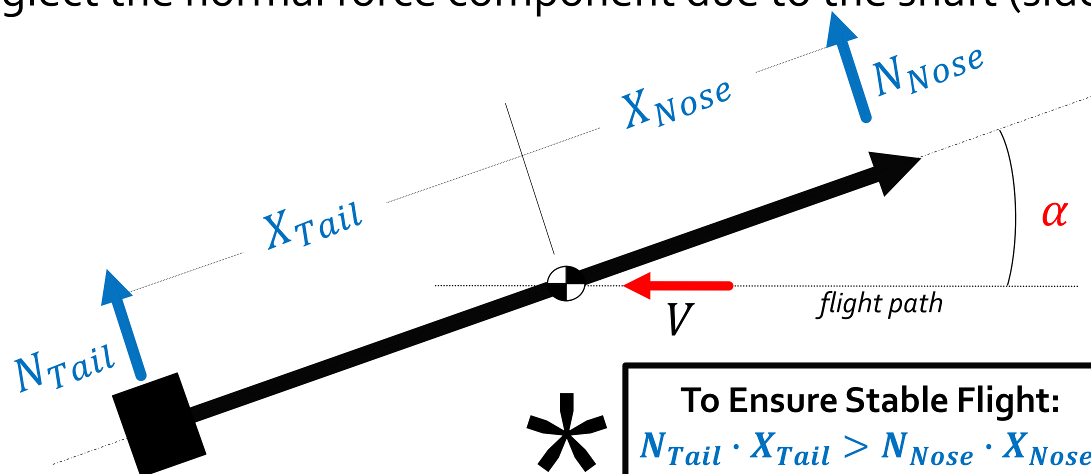
Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Instead of representing the Normal Force **N** as acting in one place on the arrow, let's instead break the Normal force into two components, where $\mathbf{N=N_{Nose} + N_{Tail}}$. Data indicate we can neglect the normal force component due to the shaft (sidebar).



* **To Ensure Stable Flight:**

$$N_{Tail} \cdot X_{Tail} > N_{Nose} \cdot X_{Nose}$$

For small angles-of-attack less than 10 degrees, the force on any cylindrical body portion is so small it can be neglected as can be seen from the following figure of reference 6.

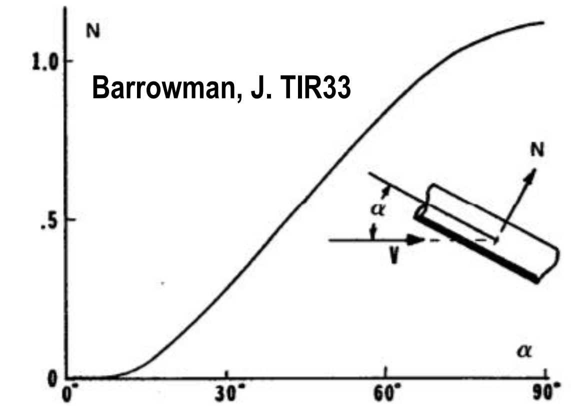


Figure 3-18 of reference 6 gives the normal force acting on circular cylinders, wires, and cables inclined to the air flow direction. This data was collected from wind tunnel tests performed primarily in the years 1918 and 1919. It is quite interesting to realize that our Space Age hobby of Model Rocketry is benefiting by engineering work done specifically to improve World War I Biplane performance.

* $N_{tail} \cdot X_{tail}$ is termed the **stabilizing moment**.
$X_{nose} \cdot N_{nose}$ is the **overturning**, or **destabilizing moment**.
The greater the difference between the two torques, the greater the stability.

Chang H. S., *Rocket Stability and the Barrowman Equations, Midwest Rocketry Club, Jan. 2020*
https://westrocketry.com/www-test/index.php/articles/rocketry-stability-and-the-barrowman-equations/

Von Braun, W. *Why Rockets Have Fins*, Popular Science, Sept. 1964
https://books.google.com/books?id=MiYDAAAAMBAJ&lpg=PP1&pg=PA68&hl=en#v=onepage&q&f=false

Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

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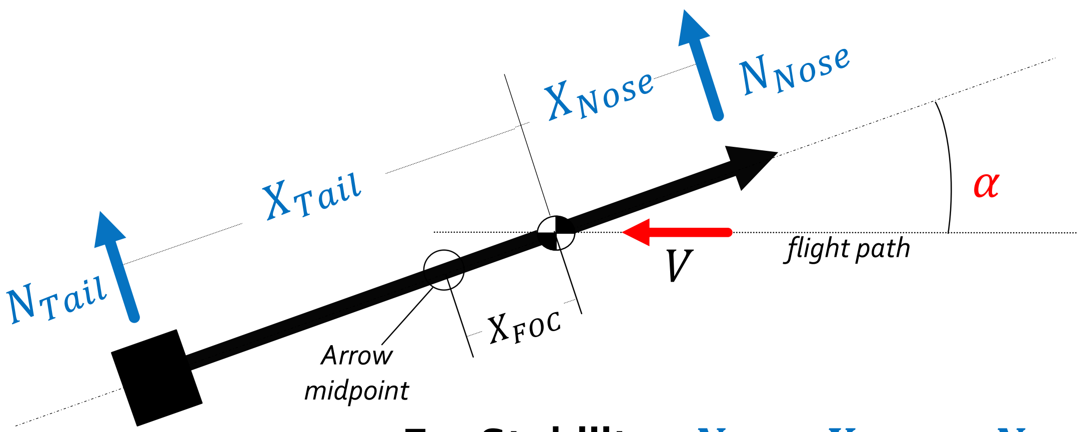
- What is the relationship between Forward Of Center (FOC) and Stability?



For Stability: $N_{Tail} \cdot X_{Tail} > N_{Nose} \cdot X_{Nose}$

- The Normal Forces do not change in position or magnitude with changing FOC
- As FOC increases, X_{tail} increases and X_{nose} decreases, therefore the stabilizing moment increases, and the overturning moment decreases.

Thus,

- The greater the FOC, the greater the stability.
- As stability increases, the magnitude of the launch disturbances are decreased.
- Therefore the arrow increasingly mimics a ballistic trajectory.

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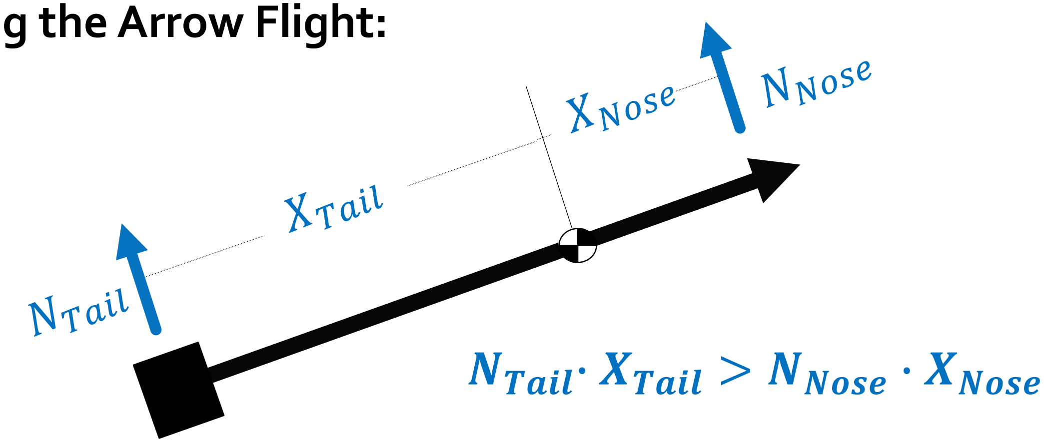
Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Implications for Tuning the Arrow Flight:



For Fixed Blade Broadheads

<u>If: You increase the size (i.e., span) of the broadhead, thereby increasing (N_{Nose})</u>

<u>Then:</u>

- Increase the FOC, thereby increasing (X_{Tail}), and reducing (X_{Nose})
- Increase the size of the fletching, which increases (N_{Tail}) but will also increase Drag
- Choose a vented broadhead, thereby reducing (N_{Nose})

<u>If: You only increase FOC, but keep the same broadhead</u>

<u>Then you can:</u>

- Reduce the fletching size, thereby reducing (N_{Tail}) which will also further increase FOC

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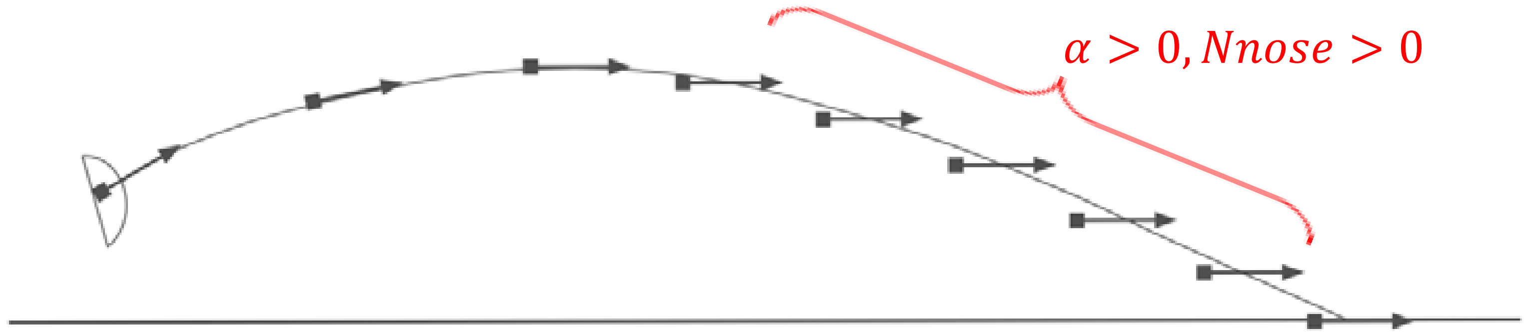
Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Comments on FOC (EFOC, UEFOC)

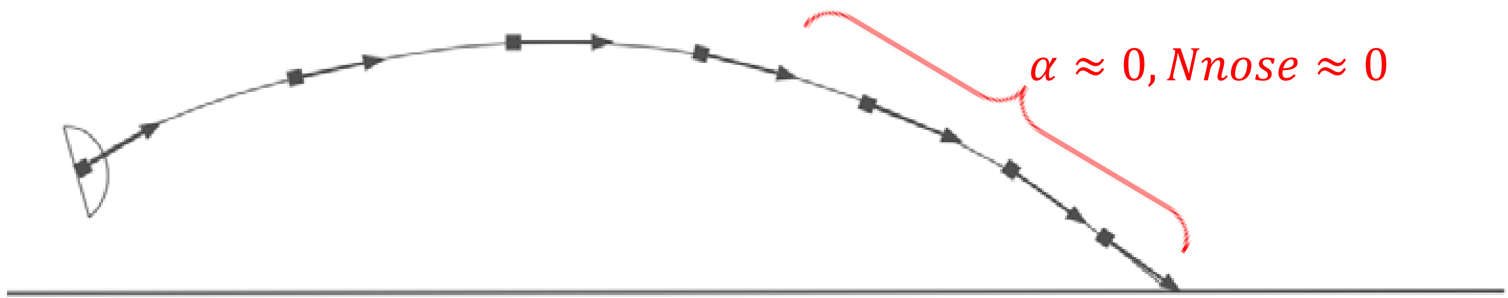
Trajectory: Here are two examples of arrow trajectory. Assume FOC is the only difference.

Marginal Stability (LOW FOC)



* trajectory exaggerated for clarity

High Stability (EFOC, UEFOC)



* trajectory exaggerated for clarity

Although an arrow with marginal stability will fly farther, the arrow with high stability will maximize the arrow's penetration capability throughout its trajectory.

[diagrams from: https://www.firenock.com/aeroflight/](https://www.firenock.com/aeroflight/)

Clarkson, S., *The Story of the Javelin – Bringing it Back Down to Earth*, The Centre for Sports Engineering Research, Sept. 2012 https://engineeringsport.co.uk/2012/09/21/the-story-of-the-javelin-bringing-it-back-down-to-earth/

Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

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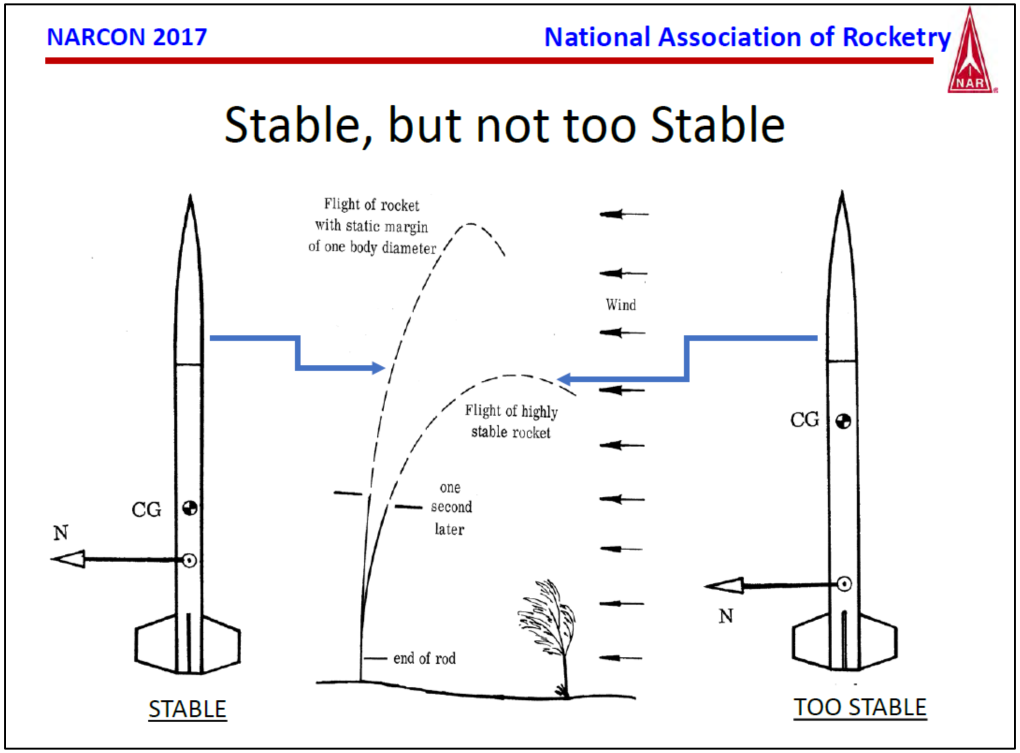
Comments on FOC (EFOC, UEFOC)

Cross Wind: The Model Rocketry community has long known that increasing stability by shifting the center of mass forward causes the rocket to turn into a crosswind.



Supposition:

This is not necessarily a detriment for arrow flight, as it is possible that slight turning into the crosswind allows compensation for lateral trajectory shift caused by the crosswind

Barrowman, J. Model Rocket Stability and Aerodynamic Equations, National Association of Rocketry, NARCON 2017

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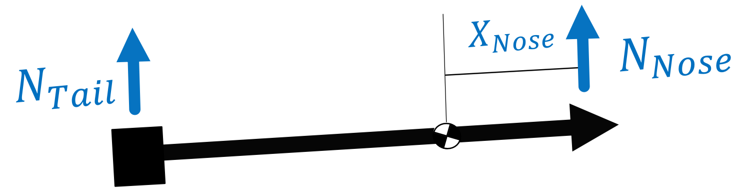
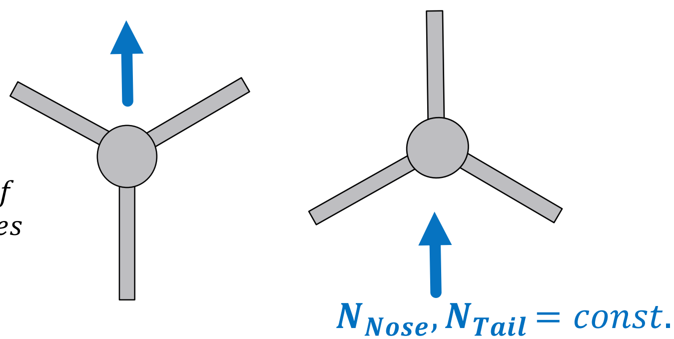
A Comment on Broadheads, Fletching and associated Normal Force:

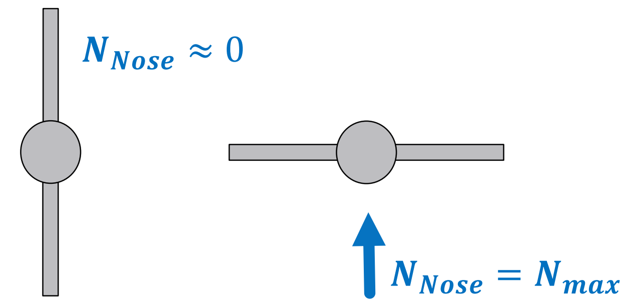


side view

$n = \textit{number of blades or vanes}$



front view of n=3-blade broadhead (or fletching)



front view of n=2-blade broadhead

if $n \geq 3$, N_{Nose}, N_{Tail} $\underline{\textit{are constant}}$ $\textit{for any roll orientation}$

if $n = 2$, N_{Nose} $\underline{\textit{varies periodically}}$ $\textit{with roll orientation}$

For a two-blade broadhead, increasing FOC reduces the effect of Normal Force (N_{Nose}) periodicity due to rolling, by shortening the overturning moment arm (X_{Nose})

Niskanen, S. *OpenRocket technical documentation*, pgs.32-33. http://openrocket.info/documentation.html

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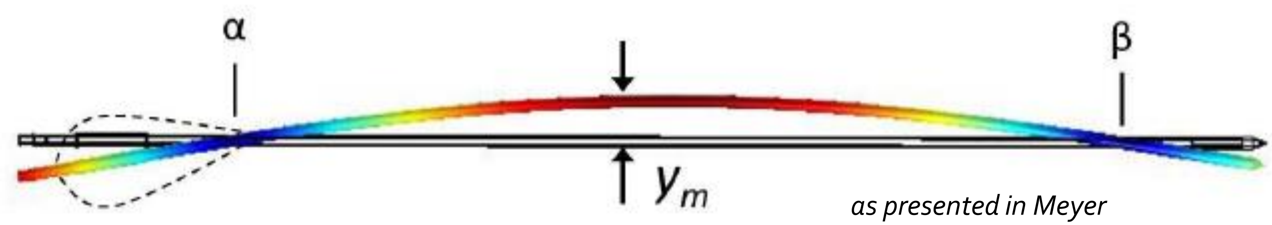
A Comment on Bending:



as presented in Meyer

Fig. 4 Fundamental mode of transverse oscillation. The coloration is related to displacement. The two nodes are labelled α and β

From Meyer, For a carbon shaft target arrow with a 96 grain point:

- Bending Frequency is ~ 86 cycles/second
- Time to damp to 37% of it's initial amplitude y_m is 1.5 seconds

Thus, for any practical hunting distance, the arrow will still be flexing at target impact

Meyer, H.O., *Application of Physics to Archery*, https://arxiv.org/abs/1511.02250

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A Comment on Bending, Normal Force (N_{Nose}), and Two-Blade Broadheads:

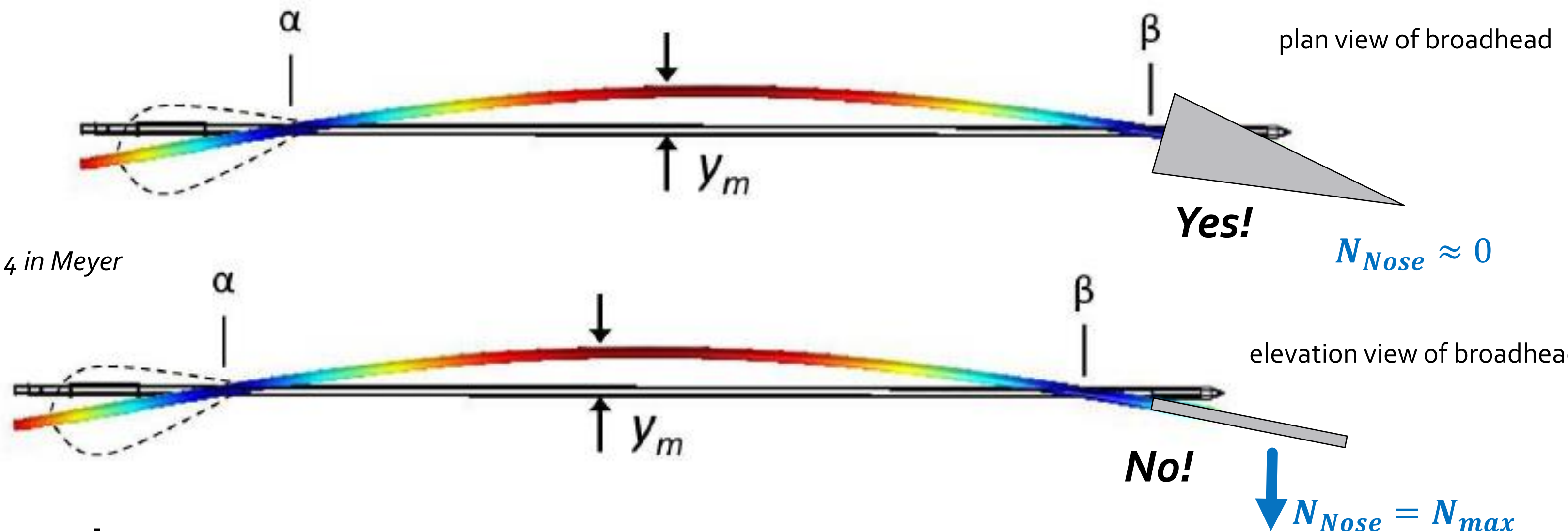

Fig. 4 in Meyer

Bench Tuning:

If: **You know the plane of the bending (orientation where the spine is weakest)**

then:

- **Align the two-blade broadhead with this plane to reduce the effect of (N_{Nose}).**
- **Orient the Nock Perpendicular to this plane as a starting point for flight tuning.**

Meyer, H.O., Application of Physics to Archery, https://arxiv.org/abs/1511.02250

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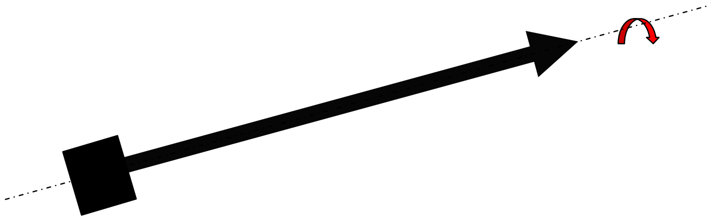
Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Comments on Rolling:



Rolling averages out the effect of mass-asymmetries and/or geometric asymmetries, and therefore reduces dispersion at the target.

- Mass asymmetries include lateral imbalances such as a internal seams, voids in the resin, runout in the shaft wall, dried blood, etc.
- Geometric asymmetries include damaged or poorly aligned fletching, bent blades bent nose, poor insert alignment, bent shaft, or bending in flight.

An issue to be aware of is roll-yaw coupling, or roll-bend coupling. If the roll rate is of the same frequency as the yaw or bend frequency, coupling of the rates can occur, leading to yaw amplification. Schematically, we can bound the problem as follows…

Park, R. *Arrows: Dynamic Behaviour Feb. 28, 2020.* https://www.bow-international.com/features/dynamic-behaviour-of-arrows/
Henstridge, G. Arrow Aerodynamics, 2019. https://www.capgo.com/FlyingSticks/book/aerodynamics.html

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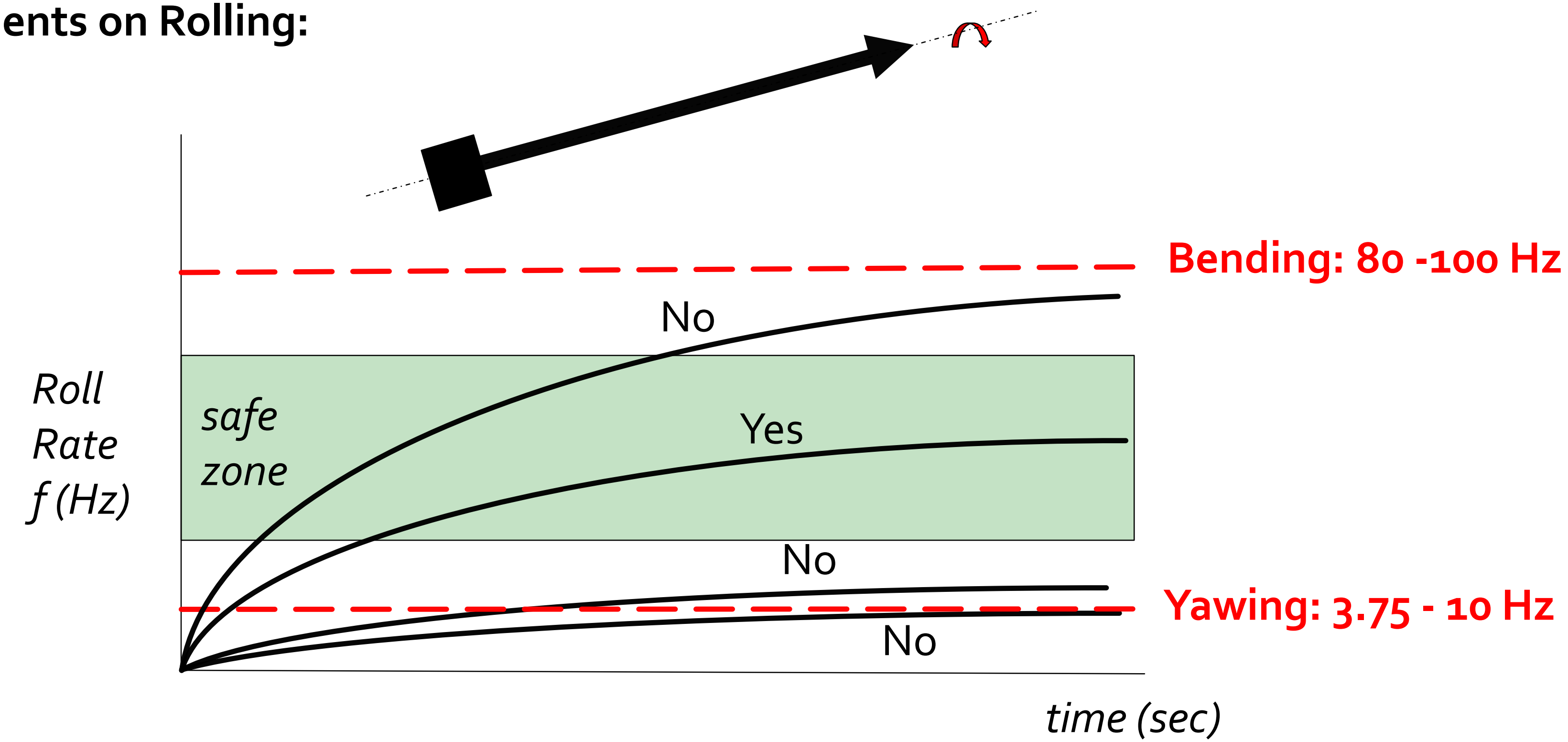
Comments on Rolling:



As a general rule: Choose your fletching to roll through the Yaw frequency quickly, but remain well below the First Bending frequency.

Mikhail, A. G., Fin Damage and Rod Eccentricity for Spin/Pitch Lock-in for Antiarmor Kinetic Energy Projectiles, ARL-TR-1442 https://apps.dtic.mil/sti/pdfs/ADA629359.pdf

Park, R. *Arrows: Dynamic Behaviour Feb. 28, 2020.* https://www.bow-international.com/features/dynamic-behaviour-of-arrows/

Henstridge, G. Arrow Aerodynamics, 2019. https://www.capgo.com/FlyingSticks/book/aerodynamics.html

SUMMARY

In this presentation, we have covered:

- **Lift and Drag, Normal Force and Axial Force Definitions**
- **Criterion for Stable Arrow Flight**
- **Ballistic Trajectory Characteristics**
- **Lateral and Trigonal Symmetry for Broadheads and Fletching**
- **Effect of FOC, EFOC, UEFOC on Trajectory and Cross-Wind**
- **Characteristics of Bending and Mitigation of its effects on Two Blade Broadheads**
- **Reasons for Rolling, and Criteria for choosing a safe Rolling Frequency**

Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Thank You!

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Questions and Discussion

Presentation Topics

Part 2: The Kinematics of Arrow Flight

Relationship Between Momentum, Kinetic Energy, and Retardation

OBJECTIVE:

What if I told you:

You can take one arrow, launch it one time, and from that single event characterize the entire flight envelope for any similar arrow of any mass launched from that bow. Here's how…

image from: http://www.clker.com/clipart-727913.html

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Relationship Between Momentum, Kinetic Energy, and Retardation

But first, we need to speak a common language:

- **Momentum** is the Quantification of Mass in Motion.
  - $M = mv\ ,\ where\ m = mass, v = velocity$

- **Kinetic Energy** is the Measure of the Capacity of Mass in Motion to do Work.
  - $KE = \frac{1}{2}mv^2,\ where\ m = mass, v = velocity$

- **Retardation** is the Change in Velocity over Distance.
  - $Ret = \frac{\Delta V}{\Delta X} = kV_0,\ where\ k = const, V_o = const.$

Each of these interrelated quantities is important in characterizing the performance of the arrow…

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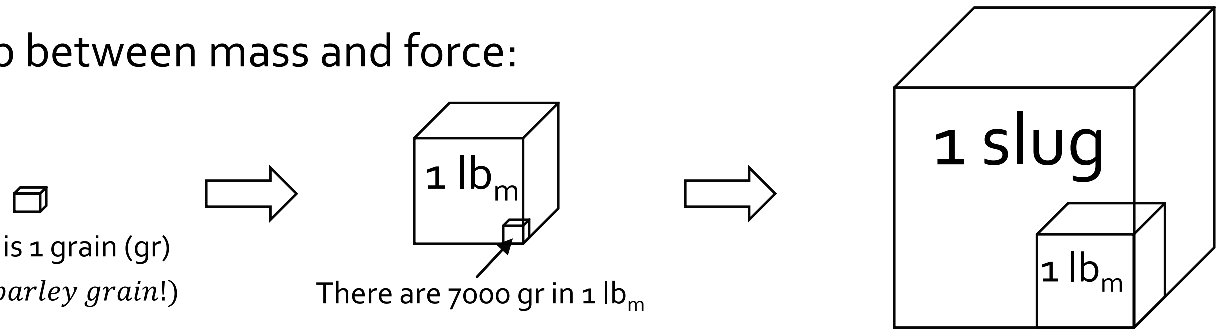
Relationship Between Momentum, Kinetic Energy, and Retardation

Sample Calculation of Kinetic Energy and Momentum

Relationship between mass and force:



$1 \text{ slug} = 1 \dfrac{\text{lb}_f\, \text{s}^2}{\text{ft}}$

This is 1 grain (gr)
(≈ 1 *barley grain*!)

There are 7000 gr in 1 lb$_m$

There are 32.174 lb$_m$ in 1 slug

A 400 gr arrow is launched at 300 fps. What is its Kinetic Energy and Momentum?

- **Kinetic Energy**
- $KE = \frac{1}{2}mv^2, = \frac{1}{2}400gr \cdot \frac{1\ lb_m}{7000\ gr} \cdot \frac{1\frac{1lb_f \cdot s^2}{ft}}{32.174\ lbm} \cdot \left(300\frac{ft}{s}\right)^2 = \boldsymbol{80\ ft \cdot lb_f}$
- **Momentum**
- $M = mv\ , = 400gr \cdot \frac{1\ lb_m}{7000\ gr} \cdot \frac{1\frac{1lb_f \cdot s^2}{ft}}{32.174\ lbm} \cdot \left(300\frac{ft}{s}\right) = \boldsymbol{0.53\ lb_f \cdot s}$

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Relationship Between Momentum, Kinetic Energy, and Retardation

Sample Calculation of Kinetic Energy and Momentum

- **Kinetic Energy**
- $KE = \frac{1}{2}mv^2, = \frac{1}{2}400gr \cdot \frac{1\ lb_m}{7000\ gr} \cdot \frac{1\frac{1 lb_f \cdot s^2}{ft}}{32.174\ lbm} \cdot \left(300\,\frac{ft}{s}\right)^2 = \mathbf{80\ ft \cdot lb_f}$

- **Momentum**
- $M = mv\ , = 400gr \cdot \frac{1\ lb_m}{7000\ gr} \cdot \frac{1\frac{1 lb_f \cdot s^2}{ft}}{32.174\ lbm} \cdot \left(300\,\frac{ft}{s}\right) = \mathbf{0.53\ lb_f \cdot s}$

Notes on the units of KE and Momentum

- The resultant units of KE here represent *WORK* done on the arrow to produce the given KE. The bow string applies a force through a distance to launch the arrow. We could have as easily chosen units of mass and velocity to represent KE.

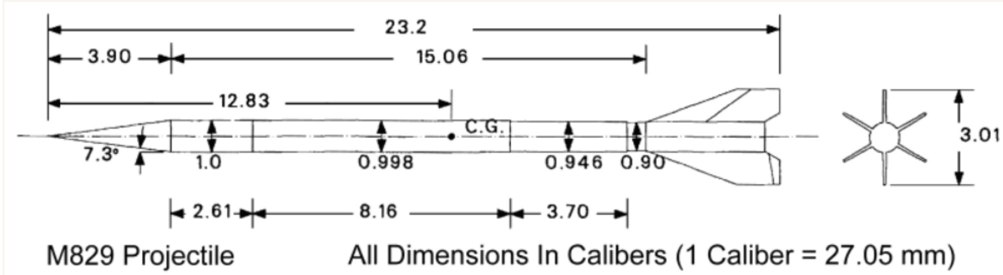
- Similarly, the resultant units of momentum indicate the *IMPULSE* (force for a given time) delivered by the bow string to the arrow, and also will indicate the impulse delivered to the target when the arrow arrives there. We could have just as easily chosen units of mass and velocity to represent the momentum of the arrow.

Relationship Between Momentum, Kinetic Energy, and Retardation

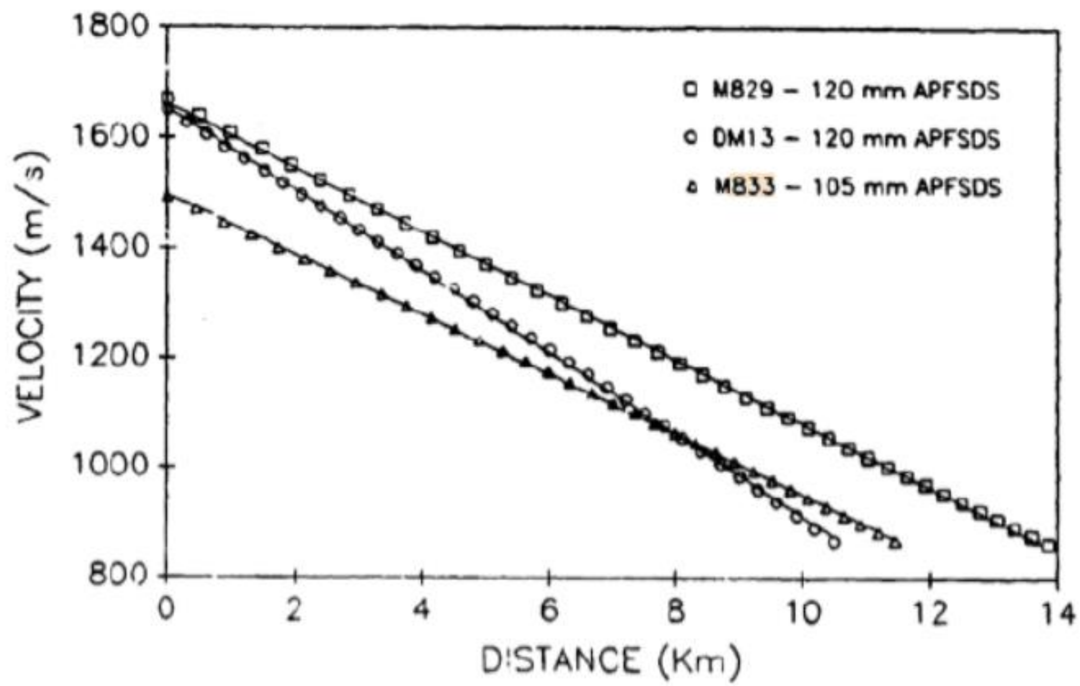
Retardation describes the rate at which an object changes velocity as a function of down range distance:

- $Ret = \frac{\Delta V}{\Delta X} = kV_0,$

$where\ k = const.,\ V_o = const.$



Firing Table Data for various Armor-Piercing Fin Stabilized Discarding Sabot (APFSDS) Supersonic Projectiles



Note here that the Retardation changes Linearly with Downrange Distance

Original Source: Ballistics Research Laboratories, Now Army Research Laboratories. Public Domain

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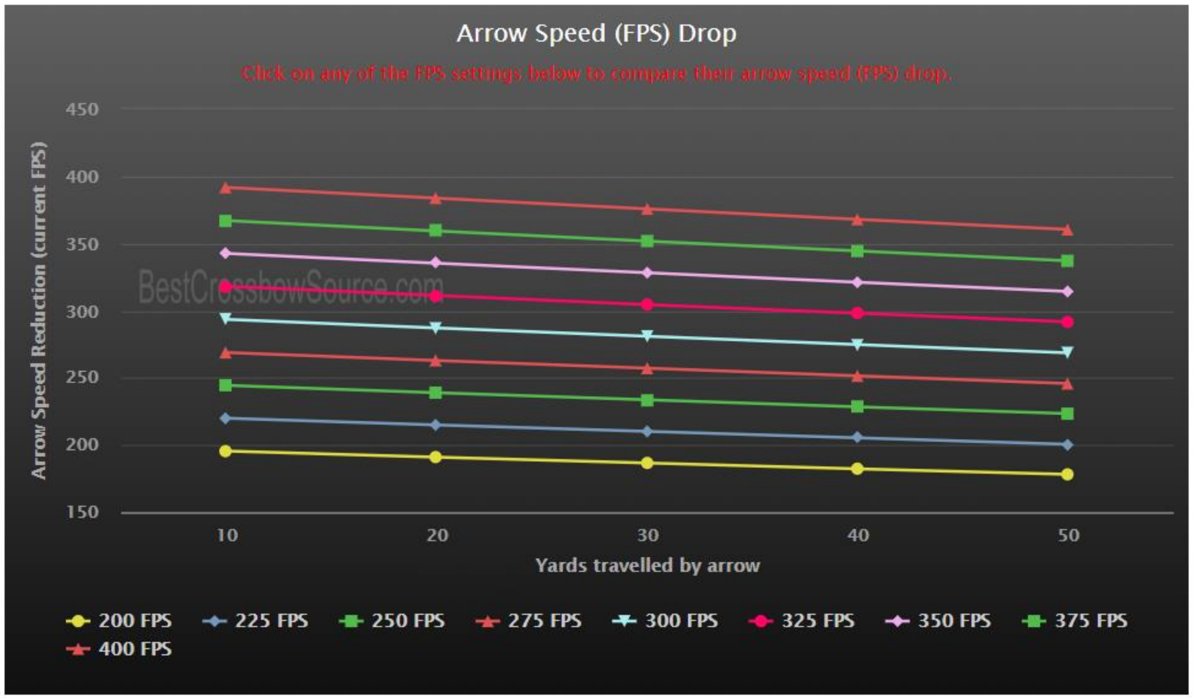
Relationship Between Momentum, Kinetic Energy, and Retardation

Is Retardation also constant for the comparatively slow subsonic arrow?



Retardation is the Change in Velocity over Distance.

$$Ret = \frac{\Delta V}{\Delta X} = kV_0, \qquad where$$

$$k = const, \ V_0 = const.$$

This chart from *bestcrossbowsource.com* was produced by firing the same arrow from 9 different crossbows through 5 chronographs stationed at 10 yard intervals.
On inspection, k (the slope) appears constant regardless of launch velocity…

http://www.bestcrossbowsource.com/arrow-speed-drop-crossbow-test-results/

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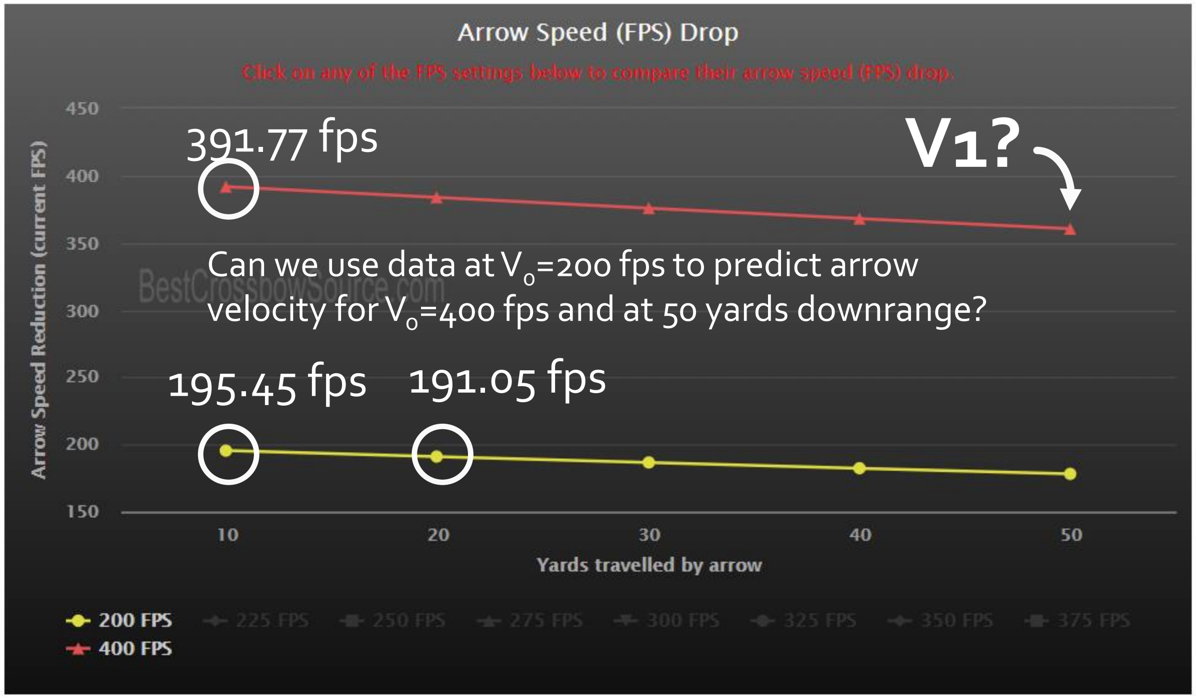
Relationship Between Momentum, Kinetic Energy, and Retardation

Let's use this dataset to check that k is invariant with launch velocity:



Retardation is the Change in Velocity over Distance.

$$Ret = \frac{\Delta V}{\Delta X} = kV_0, \qquad where$$

$$k = const,\ V_0 = const.$$

Step 1, Solve for k:

$$k = \frac{V_0 - V_1}{\Delta X \cdot V_0}$$

$$k = \frac{195.45 - 191.05}{30 \cdot 195.45}$$

$$\boldsymbol{k = 7.504E - 04}$$

Step 2, Solve for new V1:

$$V_1 = -k \cdot \Delta x \cdot V_0 + V_0$$

$$V_1 = (-7.504E - 04) \cdot 120 \cdot 391.77 + 391.77$$

$$\boldsymbol{V_1 = 356.49\ \mathbf{fps}}$$

As shown above right, the calculated value for V1 is 356.49 fps.
The measured value from a chronograph placed at 50 yards is 360.41 fps.
The difference between the measured and predicted velocities is less than 2% !
The data show Retardation is constant for a given arrow, regardless of launch velocity.
If you know *k*, you can solve for velocity at any usable down range distance.

http://www.bestcrossbowsource.com/arrow-speed-drop-crossbow-test-results/

Relationship Between Momentum, Kinetic Energy, and Retardation

What is the physical meaning of k ?

$$\boxed{Ret = \frac{\Delta V}{\Delta X} = kV_0}$$

The Retardation Equation is derived from Newton's Second Law:

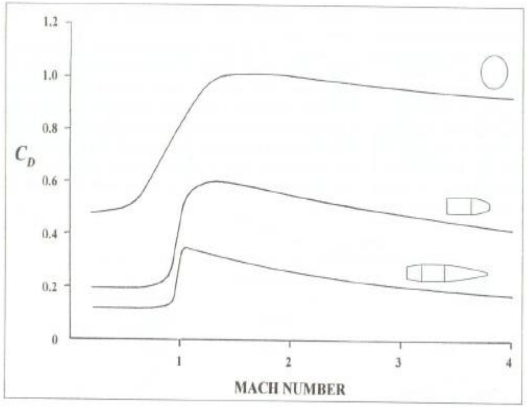
$$\boldsymbol{F = ma}$$

$$where\ F = Force, m = mass, and\ a = acceleration$$

If we cast this equation in terms of Drag (**D**) instead of a generic force (**F**) we write:

$$\boldsymbol{\frac{D}{m} = -a}$$

Drag can be re-written in terms of a drag coefficient $\boldsymbol{C_D}$. This makes sense to do because for streamlined shapes like arrows, $\boldsymbol{C_D = const.}$ below Mach 1:



then:

$$\boldsymbol{C_D = \frac{D}{\frac{1}{2}\rho V^2 A}} \quad , \quad \boldsymbol{D = C_D \tfrac{1}{2}\rho V^2 A} \, ,$$

and therefore:

$$\boldsymbol{\frac{C_D \rho A}{2m} = \frac{-a}{V^2}}, \quad \Rightarrow \quad \boldsymbol{\frac{C_D \rho A}{2m} \equiv k}$$

(continued on next slide) →

Figure from: McCoy R. *Modern Exterior Ballistics*, Schiffer Military History 1999. pg. 56

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Relationship Between Momentum, Kinetic Energy, and Retardation

What is the physical meaning of *k*?

$$k = \frac{\rho C_D A}{2m} = const.$$

$$where\ \rho = air\ density, \qquad m = mass, \qquad A = Cross - sectional\ area\ of\ shaft$$

Grouping terms:

$$\boxed{k = \frac{\rho}{2} \cdot \frac{C_D A}{m} = \frac{\rho}{2} \cdot \frac{1}{\beta}}$$

$$where\ \beta \equiv \frac{m}{C_D A}$$

i.e., the ballistic coefficient of the arrow. Note that β ratios the mass (the measure of inertia) to the **Drag** resisting the motion of the arrow

Note:

***k* is a function of the air density, arrow mass, and shaft cross-sectional area.**

To scale the Retardation of one arrow to a new geometrically similar arrow of different mass and/or area, and/or to compensate for different altitude or temperature :

$$k_{new} = k_{known} \cdot \frac{m_{known}}{m_{new}} \cdot \frac{A_{new}}{A_{known}} \cdot \frac{\rho_{new}}{\rho_{known}}$$

However, we still need a way to calculate the new arrow's V_o ...

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Relationship Between Momentum, Kinetic Energy, and Retardation

Firing the 120mm M256 gun on the US Abrams Main Battle Tank

Large Caliber Tank Guns are known to be Constant Kinetic Energy (CKE) Launchers for a wide range of launch masses. Knowing the launch mass (M_o) and muzzle velocity (V_o) for one given mass allows the estimation of (V_o) for other launch masses.

This begs the question, <u>are Compound Bows also CKE Launchers</u>? If so we can use that fact to compute the launch velocity of an arrow of any given mass.

Let's examine some data…

Relationship Between Momentum, Kinetic Energy, and Retardation

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Here are data from four modern compound bows :

Bow	Poundage	Draw Length	#Arrows Tested	Arrow Mass Range
Mathews VXR 31.5”	70	28	5	376-509
		30	5	
Prime Black 3	70.2	28	6	380-532
		30	6	
Hoyt RX-3 Ultra, #2 Cam	70.5	29.5	5	360-502
Hoyt Helix Ultra, #3 Cam	70.5	29.5	5	360-502

Data were collected from bows reviewed by Inside Out Precision Youtube Channel:
Matthews VXR 31.5”: https://www.youtube.com/watch?v=Wx4t1f4NS-I
Prime Black 3: https://www.youtube.com/watch?v=asI3SxutFXg&feature=youtu.be
Hoyt RX-3 and Hoyt Ultra: https://www.youtube.com/watch?v=nI1h8YJdrqM

Relationship Between Momentum, Kinetic Energy, and Retardation

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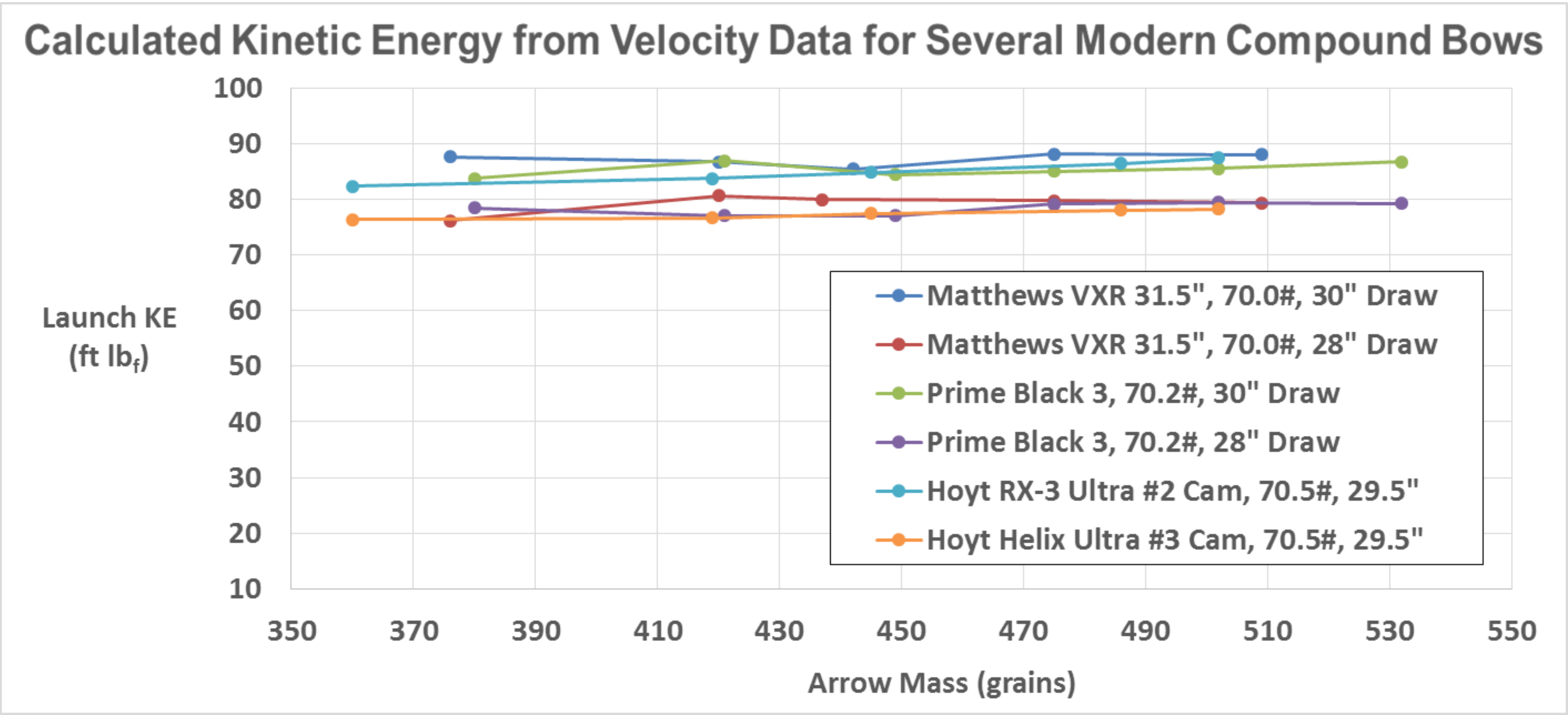
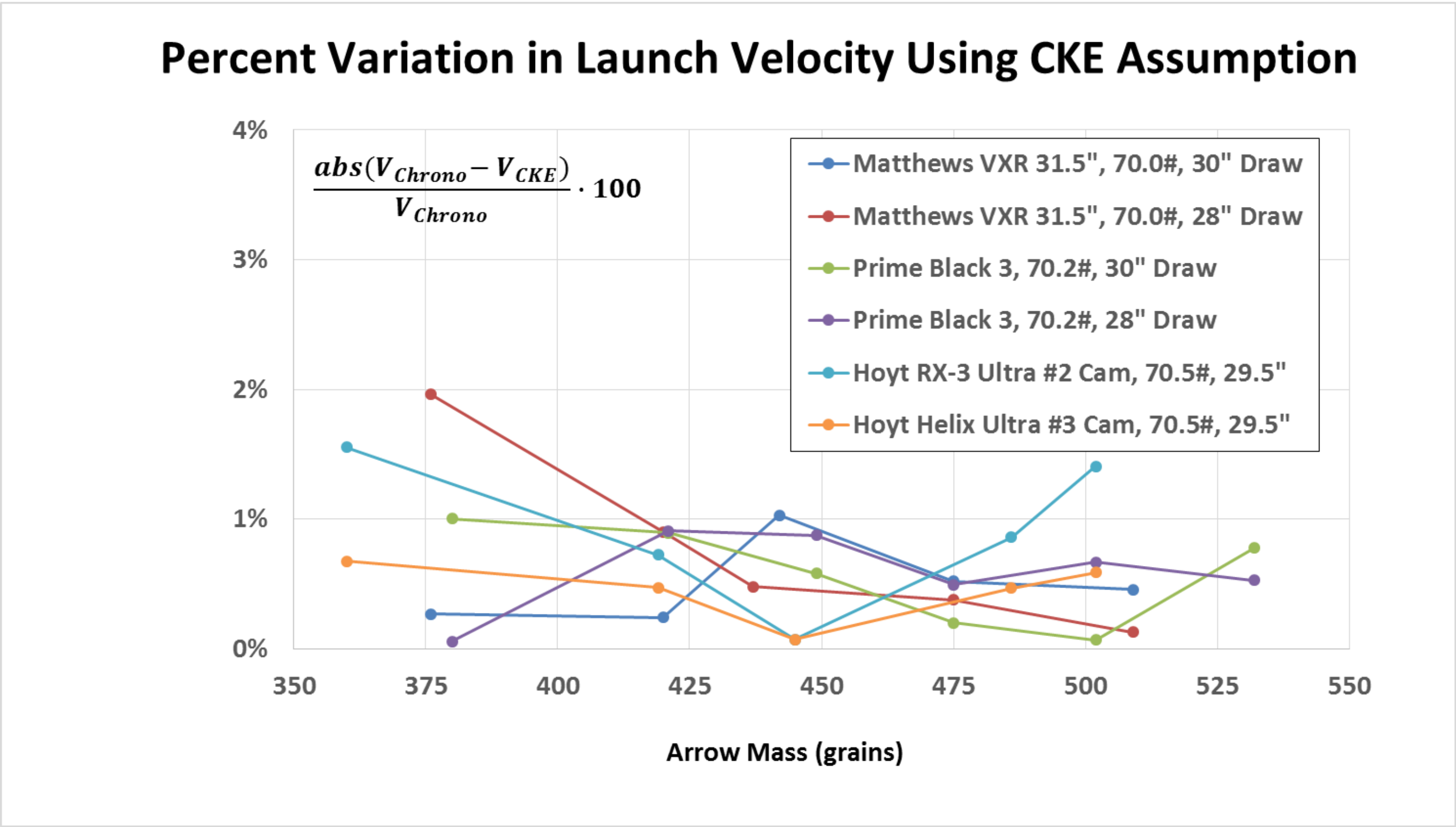


On Inspection, Kinetic Energy appears constant regardless of arrow mass.

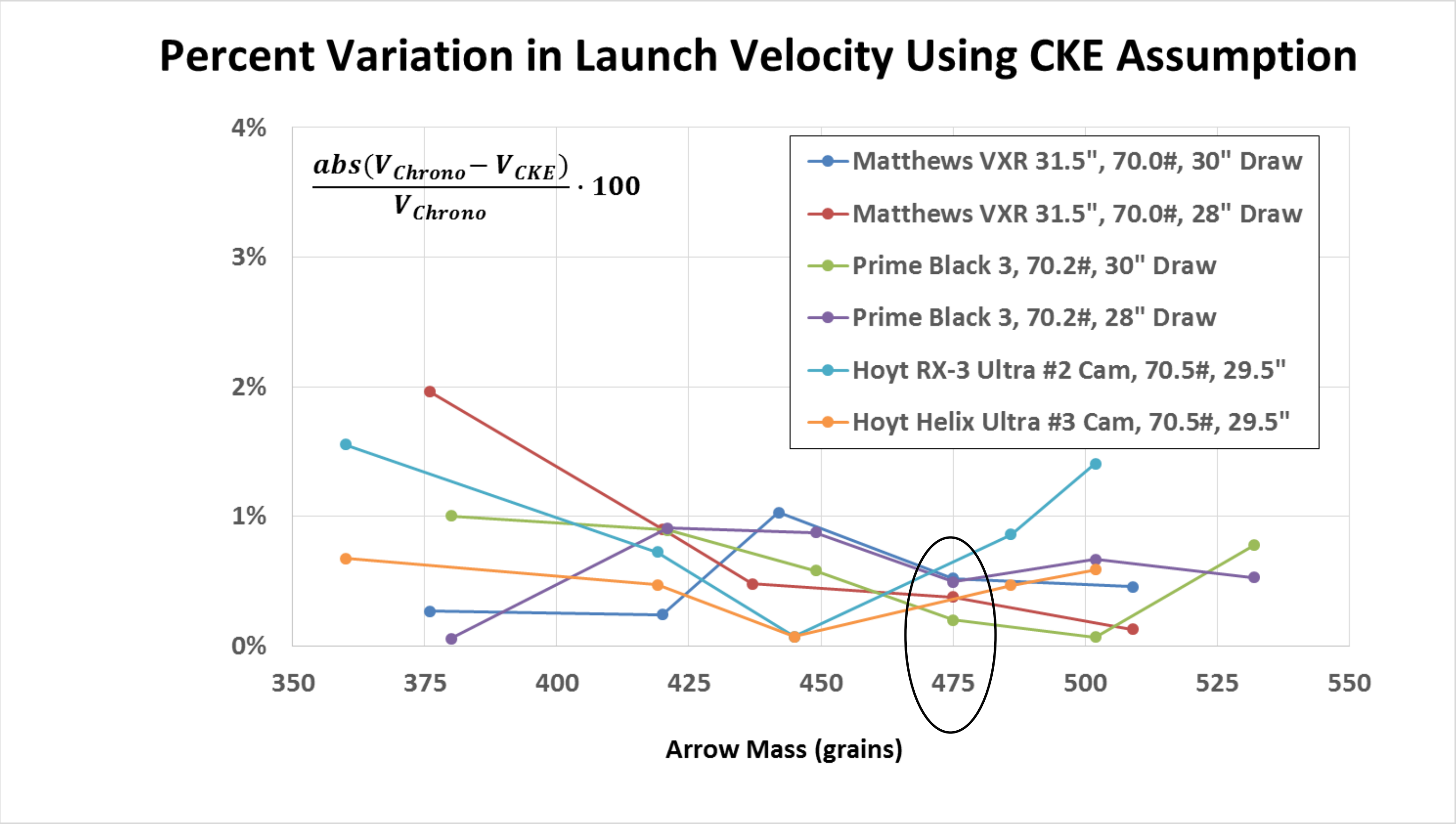
Let's check this a little closer…

Relationship Between Momentum, Kinetic Energy, and Retardation



The averaged KE value for each bow was used to back-calculate arrow velocity for each arrow mass. The results show less than 2% error in launch velocity assuming CKE. So YES, for these bows CKE is a good assumption.

Relationship Between Momentum, Kinetic Energy, and Retardation



Taken to the extreme, if you are going to determine CKE for your bow from just one shot with just one arrow, use a 475 grain arrow! ☺

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Relationship Between Momentum, Kinetic Energy, and Retardation

SUMMARIZING:

- Measuring V at two points along a given baseline arrow's trajectory, we can compute k and determine the residual Velocity, Momentum, and Kinetic Energy for that arrow at any reasonable launch velocity* and at any reasonable downrange distance.

- We can scale k by the mass ratio, area ratio, and air-density ratio to determine the down range performance of any new higher or lower mass, geometrically similar arrow (similar fletching and nose as the baseline arrow) if we can determine V_{launch} for the new arrow mass.

- For a given bow, we can determine V_{launch} for any arrow mass by knowing the CKE value of that bow.

- Now knowing V_{launch} for the new arrow, we can use the scaled value of k to predict the residual Velocity, Momentum, and Kinetic Energy of that new arrow.

*McCoy R. *Modern Exterior Ballistics*, Schiffer Military History 1999. Chapter 5, and pg. 96

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Relationship Between Momentum, Kinetic Energy, and Retardation

EXAMPLE:

Given:
Bow: Hoyt Helix Ultra, #3 Cam
Arrow 1 : 360 grain, Three fletch, Field point, 2216 shaft
Arrow 2: 650 grain, Three fletch, Field point, 2413 shaft

Determine:
The launch Velocity, Momentum, and Kinetic Energy **AND**
the residual Velocity, Momentum, and Kinetic Energy at 30 and 50 yards for both arrows

Solution:
Beginning with the 360 grain arrow, Fire the arrow once over two chronographs placed at 0 yards and 10 yards (or fire the arrow twice, with the chrono at 0 yards and then at 10 yards)

We get: V_0 = 309 fps, V_{10} = 302 fps
Solve for ***k*** :

$$k = \frac{V_0 - V_{10}}{\Delta X \cdot V_0} \quad k = \frac{309-302}{30\cdot 309} \quad \boldsymbol{k = 7.55E-04\ ft^{-1}}$$

continued →

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Relationship Between Momentum, Kinetic Energy, and Retardation

Given:
Arrow 1 : 360 grain, Three fletch, Field point, 2216 shaft
Arrow 2: 650 grain, Three fletch, Field point, 2413 shaft

We get: Vo = 309 fps, V10 = 302 fps

Calculate the Field Variables for Arrow 1 via the equation shown in the last column:

Arrow 1 360 gr	At Bow	10 Yards	30 Yards	50 Yards	Equation
Velocity (fps)	309	302	288	274	$V_x = -\boldsymbol{k} \cdot \Delta x \cdot V_0 + V_0$
Momentum ($lb_f \cdot s$)	0.494	0.483	0.460	.438	$M = mv$
Kinetic Energy ($ft \cdot lb_f$)	76.3	72.9	66.2	60.0	$KE = \frac{1}{2}mv^2$

continued →

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Relationship Between Momentum, Kinetic Energy, and Retardation

Given:
Arrow 1 : 360 grain, Three fletch, Field point, 2216 shaft
Arrow 2: 650 grain, Three fletch, Field point, 2413 shaft

Next: scale *k* for Arrow 2:

$$\boldsymbol{k_1 = 7.55E - 04\ ft^{-1}}$$

$$\boldsymbol{k_2 = k_1 \cdot \frac{m_1}{m_2} \cdot \frac{A_2}{A_1} \cdot \frac{\rho_2}{\rho_1} = k_1 \cdot \frac{360}{650} \cdot \frac{24^2}{22^2} \cdot \frac{1}{1} = 4.98E - 04\ ft^{-1}}$$

And calculate the Launch Velocity of Arrow2 from CKE:

$$\boldsymbol{V_2 = \sqrt{\frac{2KE}{m}} = \sqrt{\frac{2(76.3)}{\frac{650}{7000 \cdot 32.174}}} = 229.9\ \frac{ft}{s}}$$

continued →

Relationship Between Momentum, Kinetic Energy, and Retardation

Given:
Arrow 1 : 360 grain, Three fletch, Field point, 2216 shaft
Arrow 2: 650 grain, Three fletch, Field point, 2413 shaft

Calculate the Field Variables for Arrow 2:

Arrow 2 650 gr	At Bow	10 Yards	30 Yards	50 Yards	Equation
Velocity (fps)	230	227	220	213	$V_x = -\mathbf{k} \cdot \Delta x \cdot V_0 + V_0$
Momentum ($lb_f \cdot s$)	.664	.655	.635	.615	$M = mv$
Kinetic Energy ($\mathrm{ft} \cdot \mathrm{lb_f}$)	76.3	74.3	69.9	65.5	$KE = \frac{1}{2} mv^2$

Now that we have the calculations completed, we can compare the estimated performance of the 650 grain arrow with that of the 360 grain arrow.

continued →

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Relationship Between Momentum, Kinetic Energy, and Retardation

Arrow 1 360 gr	At Bow	10 Yards	30 Yards	50 Yards	Equation
Velocity (fps)	309	302	288	274	$V_x = -k_1 \cdot \Delta x \cdot V_0 + V_0$
Momentum $(lb_f \cdot s)$	0.494	0.483	0.460	.438	$M = mv$
Kinetic Energy $(\mathrm{ft} \cdot \mathrm{lb_f})$	76.3	72.9	66.2	60.0	$KE = \frac{1}{2}mv^2$
Time of Flight (TOF,s)	0	.098	.302	.516	McCoy*
Ratio of Sound Arrival Time to TOF	0	3.65	3.75	3.84	$Speed\ of\ Sound = \sqrt{\gamma RT}$

Arrow 2 650 gr	At Bow	10 Yards	30 Yards	50 Yards	Equation
Velocity (fps)	230	227	220	213	$V_x = -k_2 \cdot \Delta x \cdot V_0 + V_0$
Momentum $(lb_f \cdot s)$	.664	.655	.635	.615	$M = mv$
Kinetic Energy $(\mathrm{ft} \cdot \mathrm{lb_f})$	76.3	74.3	69.9	65.5	$KE = \frac{1}{2}mv^2$
Time of Flight (s)	0	.131	.400	.678	McCoy*
Ratio of Sound Arrival Time to TOF	0	4.88	4.96	5.05	$Speed\ of\ Sound = \sqrt{\gamma RT}$

*McCoy R. *Modern Exterior Ballistics*, Schiffer Military History 1999. Chapter 5, and pg. 92

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Relationship Between Momentum, Kinetic Energy, and Retardation

Several points can be made based on the resultant calculations:

- Recall that both arrows have the same Kinetic Energy when launched, as the bow is assumed to be a CKE launcher based on collected data.
- The heavier 650 gr. arrow retains more downrange Kinetic Energy than the lighter 360 gr. arrow at every downrange distance.
- The heavier 650 gr. arrow is launched with more Momentum than the lighter 360 gr. arrow, and retains more Momentum than the lighter arrow at every downrange distance
- The lighter 360 gr. arrow retains a speed advantage over the heavier 650 gr. arrow at every downrange distance
- For the lighter 360 gr. arrow, sound arrives approximately 3.8 times sooner than the arrow at every downrange distance
- For the heavier 650 gr. arrow, sound arrives approximately 5.0 times sooner than the
- arrow at every downrange distance

Although not done here, it would be of interest to experimentally examine the validity of the extrapolation process as outlined. It is expected that the analysis has a maximum error in velocity of approximately 4%. Experimental validation is left as an exercise for the interested archer.

SUMMARY

In this presentation, we have covered:

- **The Relationship between Mass and Force**
- **Sample Calculations of Kinetic Energy and Momentum with proper units**
- **Retardation and the importance of the k constant**
- **The Bow as a Constant KE Launcher**
- **Sample Calculation of Retardation and Scaling**
- **Comparison of Kinetic Energy and Momentum of a light and a heavy arrow**

Aerodynamic Characteristics of Perfect vs Actual Arrow Flight

Thank You!

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Questions and Discussion